Fall 2013 **Math 601** Name: ____________________

Quiz 9

Question 1. (10 pts)

(a) Determine whether the function $f(z) = z\bar{z} + \bar{z}$ is analytic on $\mathbb{C}$, where $z = x + iy$ and $\bar{z} = x - iy$.

Solution:

$$f(z) = x^2 + y^2 + x - iy.$$

The real part is $u(x, y) = x^2 + y^2 + x$ and the imaginary part is $v(x, y) = -y$. So

$$\frac{\partial u}{\partial x} = 2x + 1, \quad \frac{\partial v}{\partial y} = -1$$

Therefore

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$$

that is, one of the Cauchy-Riemann equations is not satisfied. So $f(z)$ is not entire.

(b) Determine whether the function $g(z) = e^{\bar{z}}$ is analytic on $\mathbb{C}$, where $\bar{z} = x - iy$.

Solution: Similarly, the real part is $u(x, y) = e^x \cos y$ and the imaginary part is $v(x, y) = -e^x \sin y$.

$$\frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial v}{\partial y} = -e^x \cos y.$$

Again,

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$$

that is, one of the Cauchy-Riemann equations is not satisfied. So $g(z)$ is not entire.

Question 2. (10 pts)

Evaluate the following integrals.

(a)

$$\int_C (z+1)dz$$

where C is the curve $y = 2x^3 - \sin(\pi x) + 1$ joining the points $(0,1)$ and $(1,3)$.

Solution: Notice that $(z+1)$ is entire and its domain is $\mathbb{C}$, which is simply connected.

$$F(z) = \frac{z^2}{2} + z$$

is an antiderivative of $z+1$. So we have

$$\int_C (z+1)dz = F(1+3i) - F(i) = \frac{(1+3i)^2}{2} + (1+3i) - (\frac{i^2}{2} + i) = \cdots$$

(b)

$$\int_C \frac{z}{z^2+4} dz$$

where C is the rectangle with vertices at ± 2, $2+i$ and $-2+i$.

Solution: Notice that

$$\frac{z}{z^2+4} = \frac{z}{(z+2i)(z-2i)}$$

which is analytic on the solid rectangle enclosed by C. So we can apply Cauchy's theorem and conclude that

$$\int_C \frac{z}{z^2+4} dz = 0$$